Random walk problem in one dimension (1D):

Let us assume a particle perform random walk in 1D.

Total no. of steps $\rightarrow N$

Length of step $\rightarrow l$

After N steps particle location $\Rightarrow x = ml$

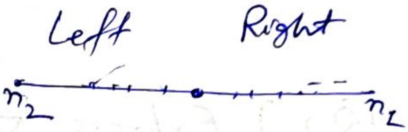
where $-N \leq m \leq N$

Next, we calculate probability of finding the particle at the position $x = ml$ after N steps. & Let it is denoted by $P_N(m)$

Note: Assumptions $\rightarrow$ Successive steps are statistically independent of each other

Let $n_1 \rightarrow$ no. of steps to right

$n_2 \rightarrow$ no. of steps to left



$$\therefore N = n_1 + n_2 \quad \text{--- (1)}$$

Net displacement $\rightarrow m = n_1 - n_2$ (measured to the right) --- (2)

$$m = n_1 - (N - n_1) = 2n_1 - N \quad \text{--- (3)}$$

From (3) if N is odd $\Rightarrow m \rightarrow$ odd

& if N is even $\rightarrow m$ is even

Let $p \rightarrow$ probability that the step is to the right

$q = 1 - p \rightarrow$ probability " " left

Now ~~probability~~ of any one given sequence of n_1 steps to the right and n_2 steps to the left is given by

$$\underbrace{(p \times p \cdots p)}_{n_1 \text{ times}} \times \underbrace{(q \times q \cdots q)}_{n_2 \text{ times}} = p^{n_1} q^{n_2} \quad \text{--- (4)}$$

No. of distinct possibilities that of total N steps n_1 to be right and n_2 are to be left is given as

$$\frac{N!}{n_1!\,n_2!} \quad \text{—— (5)}$$

Thus probability of taking n_1 steps to right & n_2 steps to left ($W_N(n_1)$) is given by Eq (4) × Eq. (5)

$$\boxed{W_N(n_1) = \frac{N!}{n_1!\,n_2!}\, p^{n_1} q^{n_2}} \quad \text{—— (6)}$$

From binomial expansion formula we know that

$$(p+q)^n = \sum_{n=0}^{N} \frac{N!}{n!\,(N-n)!}\, p^n q^{N-n} \quad \text{—— (7)}$$

Probability of finding the particle at the position $x = ml$ after N steps (n_1 to the right and n_2 steps to the left) $P_N(m)$ is same as $W_N(n_1)$ {see eqn (3)} —

$$\therefore \quad \boxed{P_N(m) = W_N(n_1)} \quad \text{—— (8)}$$

Now from (3) $n_1 = \frac{1}{2}(N+m)$

& $n_2 = \frac{1}{2}(N-m)$

Now from (6) & (8) we obtain

$$\boxed{P_N(m) = \frac{N!}{\left[\frac{1}{2}(N+m)\right]!\,\left[\frac{1}{2}(N-m)\right]!}\, p^{\left(\frac{N+m}{2}\right)} q^{\left(\frac{N-m}{2}\right)}} \quad \text{—— (9)}$$

For specific case if $p = q = \frac{1}{2}$

$$\boxed{P_N(m) = \frac{N!}{\left[\frac{1}{2}(N+m)\right]!\,\left[\frac{1}{2}(N-m)\right]!}\, p^{\left(\frac{N+m}{2}\right)} q^{\left(\frac{N-m}{2}\right)}} \quad \text{—— (10)}$$